J. Fluid Mech. (1982), *vol.* 121, *pp.* 443–463
Printed in Great Britain

The wave forces acting on a floating hemisphere undergoing forced periodic oscillations

By A. HULME
Department of Mathematics, University of Manchester, Oxford Road, Manchester M13 9PL

(Received 21 October 1981)

The object of this paper is to derive the added mass and damping coefficients associated with the periodic motions of a floating hemisphere. Two physically distinct cases are considered; namely those of heave and surge, where these nautical terms refer respectively to a vertical or horizontal oscillation of the body. Computations have been done and the values found for the various force coefficients are presented in tabulated form. A brief derivation of the long- and short-wave asymptotics of these coefficients has also been included.

1. Introduction

This paper is concerned with the calculation of those wave forces which are exerted on a floating hemispherical body due to a forced oscillatory motion of the body in the free surface of an inviscid incompressible fluid. Two canonical problems are considered; namely those of 'heave' and 'surge' motions, where these nautical terms are used to describe respectively a vertical or horizontal oscillation of the body.

The 'exact' solution to these problems, involving the radiation of waves by the body, will be found by constructing an expansion for the velocity potential in terms of infinite series of spherical harmonics, from which the relevant forces may easily be calculated. The motivation for doing this is twofold. Firstly, the solutions to these problems are of interest in their own right, since we expect the general properties of their solutions to be typical of similar problems involving non-spherical (but smooth) body geometries. Secondly, the methods are 'exact' in the sense that the numerical computations can be done to a very high precision, and the results used to determine the accuracy achieved by other methods (e.g. integral-equation or finite-element techniques) that can be used to treat more general body geometries. We shall restrict our attention to the case in which the fluid is considered to have an infinite depth; the formulation of the corresponding problems involving finite uniform depths presents few additional theoretical difficulties but considerably increases the analytical and computational complexity of the solutions. It should also be mentioned that the methods described in this paper could also be used to treat the physically distinct, but mathematically similar, problem of the diffraction of waves by a fixed hemisphere.

The formulation of problems involving floating hemispheres is analogous to that for the corresponding two-dimensional problems involving circular cylinders, as pioneered by Ursell (1949); it is perhaps for this reason that they have received comparatively little attention in the literature. An account of the heaving-hemisphere problem was first given by Havelock (1955), who used a method of solution similar to Ursell's earlier method for the circular cylinder. Havelock expressed the velocity potential, in terms of spherical polar co-ordinates, as the sum of a 'wave source' at the centre of the sphere together with an infinite series of 'wave-free' potentials; the velocity potential then satisfies all the conditions of the problem except that on the body surface, and this last condition is used to generate an infinite linear system of equations for the infinite number of unknowns appearing in the expansion of the potential.

The methods of solution adopted in this paper are essentially equivalent to Havelock's original treatment of the heaving-hemisphere problem, but a number of modifications have been made which considerably advance the analytical formulation of both the heave and surge problems and also allow a more rigorous justification of some of the important steps in the analysis. Moreover, Havelock had to use numerical quadrature in order to evaluate certain of his integrals. In contrast, it will be seen that all the integrals needed in the present work can be expressed in closed form in terms of elementary functions, and this must surely serve to increase the accuracy of the numerical calculations. In addition, this revised formulation immediately gives the long-wave asymptotic characteristics of the various force components, as we shall see in §§ 3, 4. A brief derivation of the short-wave asymptotics will also be given, and the results for heave agree with those given by Davis (1971) and Rhodes-Robinson (1971).

When presenting numerical results for the forces exerted on oscillating bodies it is usual to quote values for the added mass and damping coefficients, which measure respectively the components of force in phase with the acceleration and velocity of the body, and this convention will be followed here. The numerical results are presented in a tabulated form, rather than a graphical one, since this is perhaps the most useful format for those who would wish to compare the results produced by a more general method (e.g. integral equations) with the 'exact' results of the canonical problems as described here. For example, Kim (1965) has used an integral equation method to obtain numerical results for floating hemispheroids, in both the heave and surge modes. His results for the specific case of a hemisphere appear to be in good agreement with those presented in tables 1 and 2 of this paper.

Spherical harmonics will be used extensively throughout this work, and so it will be sensible to use a consistent definition of the associated Legendre function P_ν^μ. We shall adopt the following definition of the quantity $P_\nu^m(\cos\theta)$, where m is a non-negative integer;

$$P_\nu^m(\cos\theta) = (-1)^m (\sin\theta)^m \frac{d^m}{dx^m} P_\nu(x)\big|_{x=\cos\theta}, \tag{1.1}$$

where

$$P_\nu(x) = F(-\nu, \nu+1; 1; \tfrac{1}{2}(1-x)), \tag{1.2}$$

F being the hypergeometric function. We recall that when ν is an integer $P_\nu(x)$ is a Legendre polynomial in the variable x. Finally, we remark that, as a consequence of the definitions (1.1), (1.2),

$$P_n^m(\cos\theta) \equiv 0 \quad (m > n \geqslant 0).$$

2. The mathematical formulation of the problem

To fix ideas, let us take spherical polar co-ordinates (R, θ, ψ) as shown in figure 1. The surface of the hemispherical body is given by $R = a$, $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, $0 \leqslant \psi \leqslant 2\pi$. We assume that the surrounding fluid is inviscid, incompressible and irrotational, and this leads to a description of the fluid motion in terms of a velocity potential

$$\Phi^{(m)}(R, \theta, \psi; \tau),$$

where the superscript (m) corresponds to either a heaving motion of the body $(m = 0)$ or a surging motion $(m = 1)$. We will restrict our attention to time-harmonic oscillations of the body, of angular frequency ω, and we assume that the fluid motion has attained a 'steady state', in which case the time dependence of the problem can be removed by the introduction of a complex-valued potential $\phi^{(m)}(R, \theta, \psi)$, where

$$\Phi^{(m)}(R, \theta, \psi; \tau) = \mathscr{R}\{\phi^{(m)}(R, \theta, \psi)\, e^{-i\omega\tau}\}. \tag{2.1}$$

If we further restrict our attention to small oscillations of the body, then the potentials $\phi^{(m)}$ satisfy the usual conditions of linearized water-wave theory:

(continuity) $$\nabla^2 \phi^{(m)} = 0 \quad \text{in the fluid,} \tag{2.2}$$

(free surface) $$\left\{K + \frac{\partial}{\partial y}\right\} \phi^{(m)} = 0 \quad \text{on} \quad y = 0, \tag{2.3}$$

(radiation) $$r^{\frac{1}{2}} \left\{\frac{\partial}{\partial r} - iK\right\} \phi^{(m)} \to 0 \quad \text{as} \quad r \to \infty, \tag{2.4}$$

where $r = R \sin\theta$ and $K = \omega^2/g$. For simplicity we can take the boundary conditions on the body surface to be

$$\left\langle \frac{\partial \phi^{(m)}}{\partial R} \right\rangle = P_1^m(\cos\theta) \cos m\psi \quad (0 \leqslant \theta \leqslant \tfrac{1}{2}\pi, \quad 0 \leqslant \psi \leqslant 2\pi, \quad m = 0, 1), \tag{2.5}$$

where $\langle\ \rangle$ indicates that the value at $R = a$ is to be taken. This last condition implies that the ψ-dependence of the problem can be removed by writing

$$\phi^{(m)}(R, \theta, \psi) = \tilde{\phi}^{(m)}(R, \theta) \cos m\psi \tag{2.6}$$

and then solving for $\tilde{\phi}^{(m)}$.

The problem will initially be formulated in a way similar to Havelock's original treatment of the heaving hemisphere, that is, by expressing $\phi^{(m)}$ as the sum of a 'wave source' and an infinite series of 'wave-free' potentials. We can define a generalized mth-order wave source $\Psi_0^{(m)}$ by

$$\Psi_0^{(m)} = \cos m\psi \int_0^\infty \frac{k^{m+1}}{k-K} e^{-ky} J_m(kr)\, dk \quad (m \geqslant 0), \tag{2.7}$$

where this expression itself satisfies the conditions (2.2), (2.3); and by indenting the contour of integration to run under the simple pole at $k = K$ we also satisfy the radiation condition (2.4), since it can be shown that

$$\Psi_0^{(m)} \sim K^{m+1} \left(\frac{2\pi}{Kr}\right)^{\frac{1}{2}} \exp\{-Ky + i(Kr - \tfrac{1}{2}m\pi + \tfrac{1}{4}\pi)\} \cos m\psi \quad \text{as} \quad r \to \infty \tag{2.8}$$

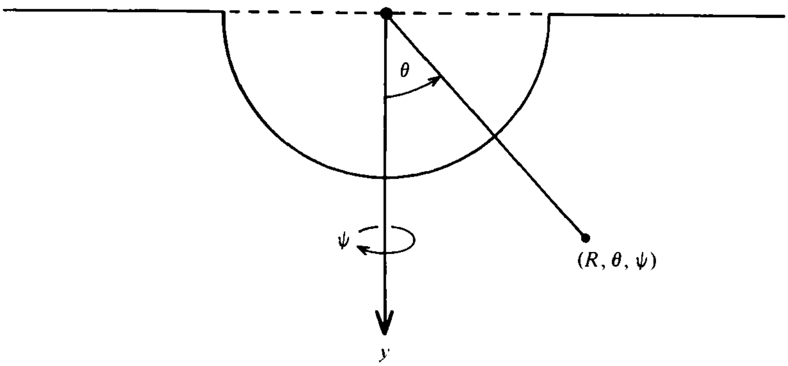

FIGURE 1. Definition sketch.

(i.e. $\Psi_0^{(m)}$ gives waves that travel radially outwards at infinity). Generalized wave-free potentials $\phi_n^{(m)}$ are given by

$$\phi_n^{(2m)} = \left\{\frac{K}{2(n-m)}\frac{1}{R^{2n}}P_{2n-1}^{2m}(\mu)+\frac{1}{R^{2n+1}}P_{2n}^{2m}(\mu)\right\}\cos 2m\psi, \tag{2.9a}$$

$$\phi_n^{(2m+1)} = \left\{\frac{K}{2(n-m)}\frac{1}{R^{2n+1}}P_{2n}^{2m+1}(\mu)+\frac{1}{R^{2n+2}}P_{2n+1}^{2m+1}(\mu)\right\}\cos (2m+1)\psi \tag{2.9b}$$

(see Thorne 1953), where for future convenience we have written $\mu = \cos\theta$. These potentials also satisfy the conditions (2.2), (2.3) and, since they clearly do not generate any waves at infinity, the $\phi_n^{(m)}$ trivially satisfy the radiation condition (2.4).

By analogy with Havelock's treatment of the heaving hemisphere problem (our $m = 0$), we consider an expansion of the form

$$\phi^{(m)} = a^{m+2}\left\{q_0^{(m)}\Psi_0^{(m)}+\sum_{t=1}^{\infty} q_t^{(m)}a^{2t}\phi_t^{(m)}\right\}. \tag{2.10}$$

By construction, $\phi^{(m)}$ satisfies all the conditions of the problem except that on the body itself; to satisfy this last condition we must have

$$P_1^m(\mu) = q_0^{(m)}\left\langle a^{m+2}\frac{\partial\tilde{\Psi}_0^{(m)}}{\partial R}\right\rangle+\sum_{t=1}^{\infty} q_t^{(m)}\left\langle a^{2t+m+2}\frac{\partial\tilde{\phi}_t^{(m)}}{\partial R}\right\rangle \quad (0 \leqslant \mu \leqslant 1). \tag{2.11}$$

Havelock now multiplies both sides of (2.11) by

$$\left\langle a^{2s+m+2}\frac{\partial\tilde{\phi}_s^{(m)}}{\partial R}\right\rangle \quad (s = 1, 2, 3, \ldots)$$

and integrates with respect to μ over $0 \leqslant \mu \leqslant 1$; this generates an infinite linear system of equations for the unknowns $\{q_s^{(m)}\}_{s\geqslant 0}$.

The main difficulty with this approach is that we need to evaluate integrals like

$$\int_0^1\left\langle a^{m+2}\frac{\partial\Psi_0^{(m)}}{\partial R}\right\rangle P_s^m(\mu)\,d\mu \tag{2.12}$$

for $s = m, m+1, \ldots$. Havelock tackled this problem by first finding an expression for the source term

$$\left\langle a^2\frac{\partial\Psi_0^{(0)}}{\partial R}\right\rangle$$

in terms of the Bessel function Y_0 and the Struve function H_0 (see his equation (24)) and then approximating the values of the integrals (2.12) by quadrature methods, but it is unlikely that this approach is suitable for large values of s because of the oscillatory nature of the Legendre function P_s^m. Nevertheless, it is widely accepted that Havelock's graphical results (for the added mass and damping coefficients of a heaving hemisphere) are qualitatively correct, and the accuracy of his calculations is to be commended, since he does not appear to have used an electronic computer. Havelock's calculations were repeated by Barakat (1962), but the results presented by the latter author seem to be in error.

Havelock only considered the case $m = 0$; the generalization of his method for any $m = 0, 1, 2, \ldots$ has recently been given by Greenhow (1980), in his investigation of interacting spherical wave-power devices. (Greenhow presented graphical results for the variation of the added-mass and damping coefficients of heaving and surging hemispheres over the range $0 \leqslant Ka \leqslant 3{\cdot}0$; the author's own calculation of these quantities extends as far as $Ka = 10{\cdot}0$, see tables 1 and 2.)

There are a number of modifications that can be made to Havelock's method which considerably improve the analytical formulation of the problem and greatly assist the numerical calculation of the quantities of interest. For clarity it is best to treat the heave and surge problems separately, but before doing so we will need the following lemma.

LEMMA. *The wave source* $\Psi_0^{(m)}$, *defined to be*

$$\Psi_0^{(m)} = \cos m\psi \int_0^\infty \frac{k^{m+1}}{k-K} e^{-ky} J_m(kr)\, dk,$$

can be expanded in terms of spherical harmonics as

$$\Psi_0^{(m)}(R,\theta,\psi) = \tilde{\Psi}_0^{(m)}(R,\theta)\cos m\psi,$$

where

$$\tilde{\Psi}_0^{(m)} = (-1)^m K^{m+1} \sum_{n=0}^{m} \frac{(n+m)!}{(KR)^{n+1}} P_n^{-m}(\mu) + \pi i\, K^{m+1} \sum_{n=m}^{\infty} \frac{(-KR)^n}{(n+m)!} P_n^m(\mu)$$
$$- K^{m+1} \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial \nu} \left\{ \frac{(KR)^\nu}{(\nu+m)!} P_\nu^m(\mu) \right\}_{\nu=n},$$

and the infinite series converge for $0 \leqslant R < \infty$.

For the proof of this lemma, see appendix A.

This generalizes an earlier result of Ursell (1963, § 3), who considered the particular case $m = 0$.

In the following work we will employ the notation

$$I\{\nu,\sigma;m\} = \int_0^1 P_\nu^m(\mu)\, P_\sigma^m(\mu)\, d\mu$$

throughout.

3. The heaving hemisphere

We will now adopt a slightly different formulation of the problem, similar to that used by Ursell (1949, 1963). We consider an expansion for $\phi^{(0)}$ of the form

$$\phi^{(0)} = C^{(0)} a^2 \left\{ \Psi_0^{(0)} + \sum_{t=1}^{\infty} p_t^{(0)} a^{2t} \phi_t^{(0)} \right\}, \tag{3.1}$$

where the (complex) constants $C^{(0)}$ and $\{p_t^{(0)}\}_{t\geqslant 1}$ are to be chosen so that the boundary condition on $R = a$ is satisfied exactly, viz

$$P_1(\mu) = C^{(0)} \left\langle a^2 \frac{\partial \Psi_0^{(0)}}{\partial R} \right\rangle - C^{(0)} \sum_{t=1}^{\infty} p_t^{(0)} \{(Ka)\, P_{2t-1}(\mu) + (2t+1)\, P_{2t}(\mu)\} \quad (0 \leqslant \mu \leqslant 1).$$

This can be rewritten as

$$\{C^{(0)}\}^{-1} P_1(\mu) = F^{(0)}(\mu, Ka) - \sum_{t=1}^{\infty} p_t^{(0)} \{(Ka)\, P_{2t-1}(\mu) + (2t+1)\, P_{2t}(\mu)\} \quad (0 \leqslant \mu \leqslant 1), \tag{3.2}$$

where

$$F^{(0)}(\mu, Ka) = \left\langle a^2 \frac{\partial \Psi_0^{(0)}}{\partial R} \right\rangle. \tag{3.3}$$

(Notice that we have made the tacit assumption that $C^{(0)}$ is non-zero – this point will be discussed later on.) Henceforth we will omit the superscript (0).

Let us now integrate (3.2) with respect to μ over $0 \leqslant \mu \leqslant 1$: this gives

$$\tfrac{1}{2} C^{-1} = \int_0^1 F(v, Ka)\, dv - (Ka) \sum_{t=1}^{\infty} p_t I\{0, 2t-1; 0\},$$

where we have used the orthogonality property

$$I\{2s, 2t; 0\} = \frac{\delta_{st}}{4s+1}. \tag{3.4}$$

We can now substitute this expression for C^{-1} into (3.2), and it follows that

$$\sum_{t=1}^{\infty} p_t [(2t+1)\, P_{2t}(\mu) + (Ka)\, \{P_{2t-1}(\mu) - 2P_1(\mu)\, I\{0, 2t-1; 0\}] = G(\mu, Ka) \quad (0 \leqslant \mu \leqslant 1), \tag{3.5}$$

where

$$G(\mu, Ka) = F(\mu, Ka) - 2P_1(\mu) \int_0^1 F(v, Ka)\, dv. \tag{3.6}$$

To solve this equation we multiply each side of (3.5) by successive elements of the complete set $\{P_{2s}(\mu)\}_{s\geqslant 1}$ and integrate over $0 \leqslant \mu \leqslant 1$. By using the result (3.4) we find that the unknowns $\{p_t\}_{t\geqslant 1}$ satisfy the infinite linear system of equations

$$p_s + (Ka) \sum_{t=1}^{\infty} p_t M_{st} = d_s \quad (s = 1, 2, 3, \ldots), \tag{3.7}$$

where

$$M_{st} = \frac{4s+1}{2s+1} \{I\{2s, 2t-1; 0\} - 2I\{2s, 1; 0\}\, I\{0, 2t-1; 0\}\} \tag{3.8}$$

(and so is independent of Ka) and

$$d_s = \frac{4s+1}{2s+1}\int_0^1 G(\mu, Ka)\, P_{2s}(\mu)\, d\mu \tag{3.9}$$

(and so is dependent on Ka). Furthermore, by using the preceding lemma we deduce that for the purpose of computation d_s can be rewritten as

$$d_s = \frac{4s+1}{2s+1}\{J\{2s, Ka\} - 2J\{0, Ka\}\, I\{2s, 1; 0\}\}, \tag{3.10}$$

where

$$\begin{aligned} J\{\sigma, Ka\} &= \int_0^1 F(\mu, Ka)\, P_\sigma(\mu)\, d\mu \\ &= -I\{\sigma, 0; 0\} - (Ka) \sum_{n=1}^{\infty} \frac{(-Ka)^n}{(n-1)!} \frac{\partial I}{\partial \nu}\{\sigma, \nu; 0\}_{\nu=n} \\ &\quad + (Ka) \sum_{n=0}^{\infty} \frac{(-Ka)^n}{n!} \{n\{\phi(n) + \pi i - \ln (Ka)\} - 1\}\, I\{\sigma, n; 0\}. \end{aligned} \tag{3.11}$$

In appendix B it is shown that any integral of the form $I\{\sigma, \nu; m\}$ can be expressed in closed form in terms of elementary functions; this must surely be an improvement on Havelock's original treatment of this problem in which he had to use numerical quadrature in order to evaluate certain of his integrals. Another feature of this approach is that the elements M_{st} are 'real' numbers, allowing the 'complex' system of equations (3.7) to be decoupled into two 'real' systems of equations.

Systems of the form (3.7) have a theory analogous to the Fredholm theory of integral equations of the second kind. If the abstract theory of compact operators is applied to the Hilbert space L_2 we obtain the latter theory; if to the Hilbert space l_2 we obtain the following version of the Fredholm alternative:

Theorem (*see Riesz* 1913, *p.* 36). *For the system*

$$x_s + \sum_{t=1}^{\infty} x_t A_{st} = c_s \quad (s = 1, 2, 3, \ldots)$$

in which

$$\sum_{s=1}^{\infty} |c_s|^2 < \infty, \quad \sum_{s=1}^{\infty}\sum_{t=1}^{\infty} |A_{st}|^2 < \infty$$

then ***either*** $\det\{\delta_{st} + A_{st}\} \neq 0$ *and for given* $\{c_s\}$ *there exists a unique solution* $\{x_s\}$, *such that*

$$\sum_{s=1}^{\infty} |x_s|^2 < \infty,$$

or $\det\{\delta_{st} + A_{st}\} = 0$ *and a solution* $\{x_s\}$ *only exists where the vector* $\{c_s\}$ *is orthogonal to all the solutions of the homogeneous transposed system.*

To obtain a bound on the $\{d_s\}$ we need the asymptotic properties of the integrals

$$I\{2s, 1; 0\}, \quad J\{2s, Ka\} \quad \text{as} \quad s \to \infty,$$

and these can be found from the results given in appendix B. It can be shown that the d_s are $O(s^{-\frac{5}{2}})$ as $s \to \infty$, and so

$$\sum_{s=1}^{\infty} s^3 |d_s|^2 < \infty,$$

which is in excess of the required condition. The results of appendix B can also be used to show that

$$\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}|M_{st}|^2<\infty,$$

and providing that the Fredholm determinant of the system (3.7) does not vanish† we are assured of the existence of a solution set $\{p_s\}_{s\geqslant 1}$, such that

$$\sum_{s=1}^{\infty}|p_s|^2<\infty.$$

This aspect of the problem has been studied by Ursell (unpublished), who showed that it is possible to establish the stronger result

$$\sum_{s=1}^{\infty}s^3|p_s|^2<\infty,$$

from which it can be shown that

$$p_s=O(s^{-\frac{2}{3}})\quad\text{as}\quad s\to\infty.$$

(To see how these results are obtained we need only refer to the corresponding proof for the surging hemisphere, which will be given in some detail in § 4.) The significance of this last result is that it shows the series for the potential (3.1) and the velocities (3.2) to be absolutely convergent for $R\geqslant a$, and so justifies all the assumptions, implicit in the analysis, that were needed to derive (3.5).

If we assume that the system (3.7) has now been solved to yield the solution set $\{p_s\}_{s\geqslant 1}$, then the remaining undetermined coefficient C is given immediately by

$$C=\tfrac{1}{2}\left(J\{0,Ka\}-(Ka)\sum_{s=1}^{\infty}p_sI\{0,2s-1;0\}\right)^{-1}, \tag{3.12}$$

and the velocity potential is now fully determined.

In a linearized theory, the excess pressure p exerted in the fluid is related to the velocity potential by

$$p=-\rho\frac{\partial\Phi}{\partial\tau}\quad(\rho=\text{fluid density}),$$

and it follows that the vertical force experienced by the body is $F^{(0)}$, where

$$F^{(0)}=\mathscr{R}\{f^{(0)}e^{-i\omega\tau}\}, \tag{3.13}$$

$$f^{(0)}=2\pi a^2\rho\omega i\int_0^1\langle\phi^{(0)}(\mu)\rangle P_1(\mu)\,d\mu. \tag{3.14}$$

We again emphasize that all the integrals needed to compute the quantity $f^{(0)}$ are known analytically. If we write the vertical force $F^{(0)}$ as

$$F^{(0)}=\tfrac{2}{3}\pi a^3\rho\omega\{B^{(0)}\cos\omega\tau-A^{(0)}\sin\omega\tau\} \tag{3.15}$$

then $A^{(0)}$ and $B^{(0)}$ are respectively the non-dimensional added-mass and damping coefficients associated with the heaving motion of the body.

† The case in which $\det\{\delta_{st}+(Ka)\,M_{st}\}=0$ can occur, at most, at a discrete set of values of Ka at which the radiated-wave amplitude vanishes at infinity; this also means that the damping coefficient, and hence C, vanish at these 'exceptional values' of Ka. At such a value of Ka, ϕ would have an expansion in terms of wave-free potentials only: the source term must be absent. I have found no such exceptional values during my computations.

To facilitate the numerical solution of the problem, the system (3.7) was truncated to a finite $N \times N$ system of equations, which was then solved exactly, using a Gaussian elimination procedure. By these means we obtain approximations to the values of the quantities $\{p_s\}_{1 \leqslant s \leqslant N}$ and C, and hence approximations to the added-mass and damping coefficients. The equations (3.7) are clearly 'diagonally dominant' for small values of Ka and we expect that solutions can be most accurately obtained in this range, although the numerical work suggests that by taking a suitably large truncated system, say $N = 50$, the method gives answers accurate to 4 decimal places for wavenumbers in the range $0 \leqslant Ka \lesssim 10$; in comparison, Havelock used only an 8×8 system of equations. The results of the calculations are shown in table 1.

The special form of the system (3.7) makes it possible to predict the asymptotic values of the coefficients $\{p_s\}_{s \geqslant 1}$ in the long-wave limit $Ka \to 0$, since it is clear that

$$p_s = d_s[1 + O\{Ka\}] \quad \text{as} \quad Ka \to 0.$$

Now (3.10), and (3.11) give expressions for the $\{d_s\}_{s \geqslant 1}$ in terms of the functions

$$(Ka)^n, \quad (Ka)^n \ln (Ka) \quad (n = 0, 1, 2, \ldots),$$

and by a careful analysis we can show that

$$\left.\begin{aligned} \mathscr{R}\{p_s\} &= 2\frac{4s+1}{2s+1} I\{2s, 1; 0\} + O\{Ka\}, \\ \mathscr{I}\{p_s\} &= \frac{\pi}{2s+1}\delta_{s1}(Ka)^3 + O\{(Ka)^4\} \end{aligned}\right\} \quad \text{as} \quad Ka \to 0.$$

The asymptotic form of the coefficient C is now given by (3.12); we find that

$$\left.\begin{aligned} \mathscr{R}\{C\} &= -\tfrac{1}{2} + O\{Ka\}, \\ \mathscr{I}\{C\} &= \tfrac{1}{4}\pi(Ka)^2 + O\{(Ka)^3\} \end{aligned}\right\} \quad \text{as} \quad Ka \to 0.$$

By using these results for $\{p_s\}_{s \geqslant 1}$ and C, and the lemma of § 2, we can now go on to deduce the asymptotic form of $f^{(0)}$, and hence that of $A^{(0)}$ and $B^{(0)}$, as $Ka \to 0$; the analysis is straightforward but laborious, and we shall merely state the important results that the long-wave asymptotic behaviours of the added-mass and damping coefficients are given by

$$\left.\begin{aligned} A^{(0)} &= L - \tfrac{3}{4}(Ka) \ln (Ka) + O\{Ka\}, \\ B^{(0)} &= \tfrac{3}{4}\pi(Ka) + O\{(Ka)^2\} \end{aligned}\right\} \quad \text{as} \quad Ka \to 0, \qquad \begin{aligned} &(3.16) \\ &(3.17) \end{aligned}$$

where

$$L = 3 \sum_{s=0}^{\infty} \frac{4s+1}{2s+1} [I\{2s, 1; 0\}]^2 = 0{\cdot}830\,951\ldots. \tag{3.18}$$

This evaluation of the constant L agrees with that given by Ursell (1963, § 4).

The behaviour of the added-mass coefficient at small values of Ka has been the subject of some debate. Havelock's original calculations show $A^{(0)}$ to be an initially increasing function of Ka, whereas Barakat (1962) claimed that $A^{(0)}$ decreased from its limiting value ($= L$) as Ka increases from zero. The result (3.16) shows that Havelock was correct. (Barakat's error was noted by Kotik & Mangulis (1962), who used the Kramers–Kronig relations to deduce long-wave asymptotics for the added masses of quite general bodies.)

Ka	Added mass $A^{(0)}$	Damping $B^{(0)}$
0	0·8310	0
0·05	0·8764	0·1036
0·1	0·8627	0·1816
0·2	0·7938	0·2793
0·3	0·7157	0·3254
0·4	0·6452	0·3410
0·5	0·5861	0·3391
0·6	0·5381	0·3271
0·7	0·4999	0·3098
0·8	0·4698	0·2899
0·9	0·4464	0·2691
1·0	0·4284	0·2484
1·2	0·4047	0·2096
1·4	0·3924	0·1756
1·6	0·3871	0·1469
1·8	0·3864	0·1229
2·0	0·3884	0·1031
2·5	0·3988	0·0674
3·0	0·4111	0·0452
4·0	0·4322	0·0219
5·0	0·4471	0·0116
6·0	0·4574	0·0066
7·0	0·4647	0·0040
8·0	0·4700	0·0026
9·0	0·4740	0·0017
10·0	0·4771	0·0012
∞	0·5	0

TABLE 1. The added-mass and damping coefficients of a heaving hemisphere

4. The surging hemisphere

The formulation of this problem follows along lines similar to those for heave motions, discussed previously, and so only a brief account need be given. The velocity potential $\phi^{(1)}$ is expressed as

$$\phi^{(1)} = C^{(1)} a^3 \left\{ \Psi_0^{(1)} + \sum_{t=1}^{\infty} p_t^{(1)} a^{2t} \phi_t^{(1)} \right\}, \tag{4.1}$$

where this satisfies all the conditions of the problem except that on the body surface, $R = a$. To satisfy this last condition we require that the coefficients $C^{(1)}$ and $\{p_t^{(1)}\}_{t \geqslant 1}$ satisfy the equation

$$\{C^{(1)}\}^{-1} P_1^1(\mu) = F^{(1)}(\mu, Ka) - \sum_{t=1}^{\infty} p_t^{(1)} \left\{ (Ka) \frac{(2t+1)}{2t} P_{2t}^1(\mu) + 2(t+1) P_{2t+1}^1(\mu) \right\} \quad (0 \leqslant \mu \leqslant 1), \tag{4.2}$$

where

$$F^{(1)}(\mu, Ka) = \left\langle a^3 \frac{\partial \tilde{\Psi}_0^{(1)}}{\partial R} \right\rangle. \tag{4.3}$$

Henceforth, we will omit the superscript (1).

We need to eliminate the unknown constant C from (4.2), and to do this we multiply both sides of the equation by $P_1^1(\mu)$ and integrate over $0 \leqslant \mu \leqslant 1$; this gives

$$\tfrac{2}{3} C^{-1} = \int_0^1 F(v, Ka) P_1^1(v)\, dv - \sum_{t=1}^{\infty} p_t \left\{ (Ka) \frac{(2t+1)}{2t} I\{1, 2t; 1\} \right\}, \tag{4.4}$$

where we have used the orthogonality result

$$I\{2s+1, 2t+1; 1\} = \frac{2(s+1)(2s+1)}{4s+3}\delta_{st}. \tag{4.5}$$

Substituting this expression for C back into (4.2), we deduce that the $\{p_t\}_{t\geqslant 1}$ satisfy the equation

$$\sum_{t=1}^{\infty} p_t\left[(Ka)\frac{(2t+1)}{2t}\{P_{2t}^1(\mu)-\tfrac{3}{2}P_1^1(\mu)\,I\{1,2t;1\}\}+2(t+1)\,P_{2t+1}^1(\mu)\right] = G(\mu, Ka) \qquad (0 \leqslant \mu \leqslant 1), \tag{4.6}$$

where

$$G(\mu, Ka) = F(\mu, Ka) - \tfrac{3}{2}P_1^1(\mu)\int_0^1 F(v, Ka)\,P_1^1(v)\,dv. \tag{4.7}$$

To generate an infinite linear system of equations for the $\{p_t\}_{t\geqslant 1}$ we successively multiply both sides of (4.6) by $P_{2s+1}^1(\mu)\,(s = 1, 2, 3, \ldots)$ and integrate over $0 \leqslant \mu \leqslant 1$. Using (4.5), we find that

$$p_s + (Ka)\sum_{t=1}^{\infty} p_t M_{st} = d_s \quad (s = 1, 2, 3, \ldots), \tag{4.8}$$

where

$$M_{st} = \frac{(4s+3)(2t+1)}{8t(2s+1)(s+1)^2}I\{2s+1, 2t; 1\} \tag{4.9}$$

(and so is independent of Ka) and

$$d_s = \frac{4s+1}{4(2s+1)(s+1)^2}\int_0^1 G(\mu, Ka)\,P_{2s+1}^1(\mu)\,d\mu \tag{4.10}$$

(and so is dependent on Ka). As before, we can use the lemma of § 2 to show that for the purposes of numerical computations d_s can be more usefully written as

$$d_s = \frac{4s+1}{4(2s+1)(s+1)^2}J\{2s+1, Ka\} \quad (s = 1, 2, 3, \ldots), \tag{4.11}$$

where

$$\begin{aligned} J\{\sigma, Ka\} &= \int_0^1 F(\mu, Ka)\,P_{2s+1}^1(\mu)\,d\mu \\ &= \int_0^1 \{(Ka)\,P_0^{-1}(\mu) + 4P_1^{-1}(\mu)\}\,P_\sigma^1(\mu)\,d\mu\,(\dagger) - (Ka)^2\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n+1)!}n\frac{\partial I}{\partial \nu}\{\sigma, \nu; 1\}_{\nu=n} \\ &\quad + (Ka)^2\sum_{n=1}^{\infty}\frac{(-Ka)^n}{(n+1)!}\{n\{\phi(n+1)+\pi i-\ln(Ka)\}-1\}\,I\{\sigma, n; 1\}. \end{aligned} \tag{4.12}$$

† By using the relation

$$P_\nu^{-m}(\mu) = \frac{(\nu-m)!}{(\nu+m)!}P_\nu^m(\mu)$$

we can deduce that

$$\int_0^1 \{(Ka)\,P_0^{-1}(\mu) + 4P_1^{-1}(\mu)\}\,P_\sigma^1(\mu)\,d\mu = (Ka)\lim_{\nu\to 0}\left[\frac{1}{\nu}I\{\sigma, \nu; 1\}\right] + 2I\{\sigma, 1; 1\},$$

where, by using the results in appendix B, the above limit can be evaluated analytically.

We will now study the system of equations (4.8) in some detail. By using the analytical expressions for the values of the integrals $I\{2s+1, \nu; 1\}$, given in appendix B, it can be shown that

$$|J\{2s+1, Ka\}| < \frac{\alpha}{s^{\frac{3}{2}}} \quad \text{for} \quad s \geqslant 1,$$

where α is a constant that depends on Ka, and so it is clear from (4.11) that d_s is $O(s^{-\frac{7}{2}})$ as $s \to \infty$, giving

$$\sum_{s=1}^{\infty} s^5 |d_s|^2 < \infty. \tag{4.13}$$

It is also shown in appendix B that

$$I\{2s+1, 2t; 1\} = \frac{(-1)^{s+t}\, t}{(s+t+1)(2s-2t+1)} \left[\frac{(2s+1)!}{4^s (s!)^2}\right] \left[\frac{(2t+1)!}{4^t (t!)^2}\right],$$

from which it can be deduced that

$$|I\{2s+1, 2t; 1\}| < \beta \frac{s^{\frac{1}{2}} t^{\frac{3}{2}}}{(s+t)\,|2s-2t+1|} \quad \text{for} \quad s, t \geqslant 1,$$

where β is a positive constant, which can be determined. It follows from (4.9) that $|M_{st}|$ has an upper bound of the form

$$|M_{st}| < \frac{t^{\frac{3}{2}}}{s^{\frac{3}{2}}} \frac{\beta}{(s+t)\,|2s-2t+1|} \quad \text{for} \quad s, t \geqslant 1. \tag{4.14}$$

If we try a direct application of the l_2 theory, as stated in § 3, we would need to establish the convergence of the double series $\Sigma\Sigma\, |M_{st}|^2$, and the result (4.14) is insufficient for this purpose. However, if we make the substitution

$$p_s = s^{-\frac{5}{2}} q_s$$

in (4.8), we obtain a new system of equations for the $\{q_s\}_{s \geqslant 1}$, namely

$$q_s + (Ka) \sum_{t=1}^{\infty} q_t\, s^{\frac{5}{2}} t^{-\frac{5}{2}} M_{st} = s^{\frac{5}{2}} d_s \quad (s = 1, 2, 3, \ldots), \tag{4.15}$$

and from (4.14) we have

$$\sum_{s=1}^{\infty} \sum_{t=1}^{\infty} |s^{\frac{5}{2}} t^{-\frac{5}{2}} M_{st}|^2 < \beta^2 \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \frac{s^2}{t^2 (s+t)^2 (2s-2t+1)^2} = \beta^2 \sum_{s=1}^{\infty} s^2 T(s), \tag{4.16}$$

where

$$T(s) = \sum_{t=1}^{\infty} \frac{1}{t^2 (s+t)^2 (2s-2t+1)^2}. \tag{4.17}$$

Now, let $[\frac{1}{2}s]$ denote the greatest integer which is such that $[\frac{1}{2}s] \leqslant \frac{1}{2}s$. The function $T(s)$ can be written schematically as

$$T(s) = \sum_{t=1}^{[\frac{1}{2}s]} + \sum_{[\frac{1}{2}s]}^{[\frac{3}{2}s]} + \sum_{[\frac{3}{2}s]}^{\infty},$$

and it can be shown that

$$\sum_{t=1}^{[\frac{1}{2}s]} < \frac{1}{s^4} \sum_{t=1}^{[\frac{1}{2}s]} \frac{1}{t^2} < \frac{1}{s^4} \sum_{t=1}^{\infty} \frac{1}{t^2} < \frac{2}{s^4},$$

$$\sum_{[\frac{1}{2}s]}^{[\frac{3}{2}s]} < \frac{\Lambda}{s^4} \sum_{[\frac{1}{2}s]}^{[\frac{3}{2}s]} \frac{1}{(2s-2t+1)^2} < \frac{2\Lambda}{s^4} \sum_{t=1}^{\infty} \frac{1}{t^2} < \frac{4\Lambda}{s^4},$$

$$\sum_{[\frac{3}{2}s]}^{\infty} < \frac{1}{s^4} \sum_{[\frac{3}{2}s]}^{\infty} \frac{1}{t^2} < \frac{1}{s^4} \sum_{t=1}^{\infty} \frac{1}{t^2} < \frac{2}{s^4},$$

where Λ is a (determinable) positive constant. Hence

$$\sum_{s=1}^{\infty}\sum_{t=1}^{\infty}|s^{\frac{5}{2}}t^{-\frac{5}{2}}M_{st}|^2 < \beta^2\sum_{s=1}^{\infty}s^2T(s) < 4(1+\Lambda)\,\beta^2\sum_{s=1}^{\infty}\frac{1}{s^2} < \infty. \tag{4.18}$$

The l_2 theory for a system of equations like (4.15), and for values of Ka for which†

$$\det\{\delta_{st}+(Ka)\,s^{\frac{5}{2}}t^{-\frac{5}{2}}M_{st}\} \neq 0,$$

now gives that there exists a solution $\{q_s\}_{s\geqslant 1}$, where

$$\sum_{s=1}^{\infty}|q_s|^2 < \infty.$$

We have thus established the result

$$\sum_{s=1}^{\infty}s^5|p_s|^2 < \infty. \tag{4.19}$$

By reference to (4.15) we see that

$$p_s = d_s-(Ka)\,s^{-\frac{5}{2}}\sum_{t=1}^{\infty}(t^{\frac{5}{2}}p_t)\,s^{\frac{5}{2}}t^{-\frac{5}{2}}M_{st},$$

and so

$$|p_s| \leqslant O(s^{-\frac{7}{2}})+O(s^{-\frac{5}{2}})\left[\sum_{t=1}^{\infty}t^5|p_t|^2\right]^{\frac{1}{2}}\left[\sum_{t=1}^{\infty}|s^{\frac{5}{2}}t^{-\frac{5}{2}}M_{st}|^2\right]^{\frac{1}{2}}$$

(Schwarz's inequality)

$$= O(s^{-\frac{7}{2}}) \quad \text{by (4.14) and (4.19).}$$

This last result shows that the series for the potential and for the fluid velocities are absolutely convergent for $R \geqslant a$, and so justifies all the assumptions implicit in the preceding analysis.

The computational techniques used to obtain approximate numerical solutions to the heaving-hemisphere problem can also be used to treat the surging-hemisphere problem, since both of these can ultimately be reduced to the problem of finding the solution to an infinite system of linear equations in an infinite number of unknowns. Once approximations to the values of $\{p_t^{(1)}\}_{t\geqslant 1}$ and $C^{(1)}$ have been obtained, the horizontal force $F^{(1)}$ experienced by the body can be calculated from $\phi^{(1)}$ in the usual way. We can again describe the force $F^{(1)}$ in terms of dimensionless coefficients $A^{(1)}$, $B^{(1)}$:

$$F^{(1)} = \tfrac{2}{3}\pi a^3\rho\omega\{B^{(1)}\cos\omega\tau - A^{(1)}\sin\omega\tau\}, \tag{4.20}$$

where $A^{(1)}$ and $B^{(1)}$ are respectively the non-dimensional added-mass and damping coefficients associated with the surging motion of the hemisphere. Table 2 gives values for these coefficients over the range $0 \leqslant Ka \leqslant 10$, and the calculations are believed to be accurate to 4 places of decimals. By an argument similar to that outlined at the end of § 3, we can show that in the long-wave limit $Ka \to 0$ the added-mass and damping coefficients are given by

$$\left.\begin{aligned} A^{(1)} &= \tfrac{1}{2}+\tfrac{3}{16}(Ka)+O\{(Ka)^2\}, \quad (4.21)\\ B^{(1)} &= \tfrac{3}{8}\pi(Ka)^3+O\{(Ka)^4\} \quad (4.22)\end{aligned}\right\} \text{ as } Ka \to 0.$$

† Again, no 'exceptional values' of Ka were encountered during the computations.

Ka	Added mass $A^{(1)}$	Damping $B^{(1)}$
0	0·5	0
0·1	0·5223	0·0011
0·2	0·5515	0·0082
0·3	0·5848	0·0255
0·4	0·6175	0·0557
0·5	0·6439	0·0987
0·6	0·6586	0·1516
0·7	0·6582	0·2092
0·8	0·6421	0·2653
0·9	0·6127	0·3145
1·0	0·5740	0·3535
1·2	0·4860	0·3978
1·4	0·4038	0·4061
1·6	0·3371	0·3929
1·8	0·2865	0·3695
2·0	0·2493	0·3424
2·5	0·1951	0·2769
3·0	0·1720	0·2237
3·5	0·1634	0·1826
4·0	0·1620	0·1511
4·5	0·1641	0·1266
5·0	0·1679	0·1073
6·0	0·1772	0·0794
7·0	0·1865	0·0608
8·0	0·1949	0·0479
9·0	0·2022	0·0386
10·0	0·2085	0·0317
∞	0·2732	0

TABLE 2. The added-mass and damping coefficients of a surging hemisphere

5. Short-wave asymptotics of the added-mass and damping coefficients

In the preceding work we have expressed the velocity potentials associated with the heave and surge motions of the floating hemisphere in terms of infinite series of spherical harmonics, and we have seen that both problems ultimately reduce to that of finding the solution to an infinite linear system of equations in an infinite number of unknowns. This approach is found to be very successful for slow oscillations of the body (i.e. small Ka), but for high frequencies these infinite systems of equations are 'ill-conditioned' in the sense that as Ka increases we must solve bigger and bigger finite (i.e. truncated) systems of equations in order to calculate the added-mass and damping coefficients to a specified accuracy. However, computational experience suggests that if we solve a 50×50 system of equations we can obtain results accurate to at least 4 decimal places, for wavenumbers up to $Ka \approx 10$.

For large values of Ka it is more appropriate to formulate the problems in terms of integral equations whose kernels become 'small' as $Ka \to \infty$; a description of this method, as applied to a floating cylinder, has been given in a classical paper by Ursell (1953). This procedure has been used by Davis (1971) to give short-wave asymptotic results for the heaving hemisphere, and a similar approach could be adopted for the surge case although a rigorous treatment would involve a significant amount of mathematical labour; for this reason we will give an alternative derivation of these results, based on arguments which are physically plausible rather than mathematically precise. It will again be useful to treat heave and surge motions separately.

Heave

We follow an argument given by Ursell (1954) in connection with the motion of a floating cylinder. As $Ka \to \infty$ the free-surface condition tends formally to $\phi = 0$, and we therefore expect that the heave potential $\phi^{(0)}$ tends to the limit

$$\phi^{(0)} = -\frac{1}{2}\frac{a^3}{R^2}P_1(\mu) \quad (\mu = \cos\theta) \quad \text{as} \quad Ka \to \infty,$$

everywhere except in a thin surface layer of thickness $O(1/K)$ where there are waves. With this in mind, let us define a potential $\Lambda^{(0)}$ by

$$\phi^{(0)} = -\tfrac{1}{2}a^3\left[\frac{1}{R^2}P_1(\mu)+\frac{2}{K}\frac{1}{R^3}P_2(\mu)\right]+\frac{1}{Ka}\Lambda^{(0)}, \tag{5.1}$$

where the term in the brackets is the wave-free potential $2\phi_1^{(0)}/K$ – see (2.9a). The boundary condition on the sphere is

$$\left\langle\frac{\partial\phi^{(0)}}{\partial R}\right\rangle = P_1(\mu) \quad (0 \leqslant \mu \leqslant 1),$$

and so from (5.1) we must have

$$\begin{aligned}\left\langle\frac{\partial\Lambda^{(0)}}{\partial R}\right\rangle &= -3P_2(\mu) \quad (0 \leqslant \mu \leqslant 1) \\ &\approx \tfrac{3}{2} \quad \text{near} \quad \mu = 0, \quad \text{the free surface.}\end{aligned} \tag{5.2}$$

We have thus transformed the problem into one in which the radial fluid velocity does not vanish on the free surface.

To calculate the damping coefficient of the heaving hemisphere we need only consider the waves generated by the term $(Ka)^{-1}\Lambda^{(0)}$. For large Ka, these waves are confined to a thin surface layer, and the result (5.2) suggests that, to a first approximation, these resemble the waves generated by a vertical circular cylinder of radius a oscillating with a radial velocity of magnitude $\frac{3}{2}$ near the free surface. The latter problem has been solved by Havelock (1929, § 6) and by reference to his work we can show that

$$\begin{aligned}\Lambda^{(0)} &\sim 2e^{-Ky}\frac{H_0^{(1)}(Kr)}{H_0^{(1)}(Ka)}\int_0^\infty [\tfrac{3}{2}]\, e^{-K\nu}d\nu \quad (r = R\sin\theta) \\ &= \frac{3}{K}e^{-Ky}\frac{H_0^{(1)}(Kr)}{H_0^{(1)}(Ka)} \quad \text{as} \quad r \to \infty.\end{aligned} \tag{5.3}$$

When Kr and Ka are both large we have, from the asymptotic expression for the Hankel function $H_0^{(1)}$,

$$\Lambda^{(0)} \sim -\frac{3i}{K}e^{-Ky}\,e^{iK(r-a)}\left(\frac{a}{r}\right)^{\frac{1}{2}}, \tag{5.4}$$

and this agrees with the result given by Leppington (1973), who used the method of matched asymptotic expansions. Since

$$\phi^{(0)} \sim \frac{1}{Ka}\Lambda^{(0)} \quad \text{as} \quad r \to \infty,$$

we deduce that the (time-averaged) energy flux at infinity is asymptotic to

$$\rho\frac{9\pi\omega a^3}{2(Ka)^4} \quad \text{as} \quad Ka \to \infty. \tag{5.5}$$

Now, the (time-averaged) rate of work of the body on the fluid is

$$\tfrac{1}{3}\pi a^3 \rho\omega B^{(0)}, \tag{5.6}$$

where $B^{(0)}$ is the dimensionless damping coefficient of the heaving hemisphere; the quantities (5.5) and (5.6) must be equal, and so we deduce the asymptotic result

$$B^{(0)} \sim \frac{27}{2(Ka)^4} \quad \text{as} \quad Ka \to \infty. \tag{5.7}$$

To find the short-wave behaviour of the added-mass coefficient we again study the potential $\Lambda^{(0)}$. As $Ka \to \infty$ it seems reasonable to assume that $\Lambda^{(0)}$ tends to a limiting value that corresponds to the boundary condition $\Lambda^{(0)} = 0$ on $y = 0$, except in a thin surface layer where the pressure exerted on the body will contribute little to the total vertical force; we can thus ignore the effect of this surface discrepancy. We need to find a potential $\Lambda^{(0)}$ such that

$$\Lambda^{(0)} = 0 \quad \text{on} \quad \mu = 0, \quad \text{and} \quad \left\langle \frac{\partial \Lambda^{(0)}}{\partial R} \right\rangle = -3P_2(\mu) \quad \text{over} \quad 0 < \mu \leqslant 1.$$

Let us try

$$\Lambda^{(0)} = \sum_{n=1}^{\infty} c_n \frac{a^{2n+1}}{R^{2n}} P_{2n-1}(\mu);$$

by construction this satisfies the condition at $\mu = 0$, and if we take

$$c_n = \frac{3(4n-1)}{2n} I\{2n-1, 2; 0\}$$

we also satisfy the condition over the body surface, $0 < \mu \leqslant 1$. We can now deduce from the original expression for $\phi^{(0)}$ in (5.1) that the added mass of the body is given by

$$\begin{aligned} A^{(0)} &= -\frac{3}{a}\int_0^1 \langle \phi^{(0)}(\mu)\rangle P_1(\mu)\, d\mu, \\ &\sim 3\left[\tfrac{1}{2}I\{1,1;0\} - \frac{1}{Ka}\{c_1 I\{1,1;0\} - I\{2,1;0\}\}\right], \end{aligned}$$

i.e.

$$A^{(0)} \sim \frac{1}{2} - \frac{3}{16}\frac{1}{Ka} \quad \text{as} \quad Ka \to \infty. \tag{5.8}$$

The results (5.7), (5.8) were first stated by Ursell (1957), and subsequently re-derived by Davis (1971) and Rhodes-Robinson (1971).

Surge

To derive the short-wave asymptotics of the surge-damping coefficient $B^{(1)}$ we follow almost the same argument as before; in surge, the normal velocity on the hemisphere is

$$\left\langle \frac{\partial \phi^{(1)}}{\partial R} \right\rangle = P_1^1(\mu)\cos\psi$$
$$\approx -\cos\psi \quad \text{near} \quad \mu = 0, \quad \text{the free surface.}$$

(Notice that in this case the radial velocity does not vanish everywhere on the free surface, and so no preliminary transformation is needed.) To a first approximation we expect that, at a distance from the body, the waves due to $\phi^{(1)}$ resemble those due to an oscillating vertical cylinder of radius a, whose radial velocity near the free surface is given by $-\cos\psi$. By analogy with Havelock (1929) we deduce that

$$\phi^{(1)} \sim -2\,e^{-Ky}\frac{H_1^{(1)}(Kr)}{H_1^{(1)'}(Ka)}\cos\psi\int_0^\infty [1]\,e^{-K\nu}\,d\nu$$
$$= \frac{-2}{K}e^{-Ky}\frac{H_1^{(1)}(Kr)}{H_1^{(1)'}(Ka)}\cos\psi \quad \text{as} \quad r\to\infty.$$

The argument now proceeds exactly as before, and we need only state the result:

$$B^{(1)} \sim \frac{3}{(Ka)^2} \quad \text{as} \quad Ka\to\infty. \tag{5.9}$$

To find the short-wave behaviour of the added-mass coefficient $A^{(1)}$ we might consider an expansion of the form

$$\phi^{(1)} = \sum_{n=1}^{\infty} a^{2n+2}\phi_n^{(1)}\{\alpha_n + (Ka)^{-1}\beta_n + (Ka)^{-2}\gamma_n + \ldots\}, \tag{5.10}$$

where $\phi_n^{(1)}$ is the wave-free potential:

$$\phi_n^{(1)} = \left\{\frac{1}{R^{2n+1}}P_{2n}^1(\mu) + \frac{2n}{K}\frac{1}{R^{2n+2}}P_{2n+1}^1(\mu)\right\}\cos\psi.$$

By construction, $\phi^{(1)}$ satisfies the free-surface condition exactly, and to satisfy the boundary condition on the body we must have

$$P_1^1(\mu) = -\sum_{n=1}^{\infty}\left\{(2n+1)P_{2n}^1(\mu) + \frac{4n(n+1)}{Ka}P_{2n+1}^1(\mu)\right\}\left\{\alpha_n + \frac{\beta_n}{(Ka)} + \frac{\gamma_n}{(Ka)^2} + \ldots\right\}$$
$$(0<\mu\leqslant 1). \tag{5.11}$$

To solve this equation we can multiply both sides by $P_{2m}^1(\mu)$ $(m = 1, 2, \ldots)$, and integrate over $0<\mu\leqslant 1$; we would deduce that by taking

$$\alpha_n = -\frac{1}{2n+1}I\{2n, 1; 1\}\,[I\{2n, 2n; 1\}]^{-1}, \tag{5.12}$$

we satisfy the boundary condition (5.11) to within $O\{(Ka)^{-1}\}$, and that, by taking

$$\beta_n = -[(2n+1)\,I\{2n, 2n; 1\}]^{-1}\sum_{m=1}^{\infty} 4m(m+1)\,I\{2m+1, 2n; 1\}\,\alpha_m, \tag{5.13}$$

we satisfy the boundary condition (5.11) to within $O\{(Ka)^{-2}\}$.

If we now use these results to calculate the added-mass coefficient we would find that

$$A^{(1)} \sim C_1 - \frac{C_2}{Ka} \quad \text{as} \quad Ka \to \infty, \tag{5.14}$$

where

$$C_1 = -\frac{3}{2}\sum_{n=1}^{\infty} \alpha_n I\{2n, 1; 1\}, \tag{5.15}$$

$$C_2 = +\frac{3}{2}\sum_{n=1}^{\infty} \beta_n I\{2n, 1; 1\}. \tag{5.16}$$

The terms in the series for C_1 decay as $1/n^3$, and computations have shown that $C_1 = 0{\cdot}273239\ldots$. The series for C_2 also converges, but here the terms only decay as fast as $(\log n)/n^2$ and the result (5.16) is of little use computationally.

I wish to emphasize that this derivation of the asymptotic form of $A^{(1)}$ is suggestive rather than conclusive, although I do believe the result (5.14) to be correct.

I am grateful to Professor F. Ursell, F.R.S., for allowing me to refer to private notes on the subject of the heaving hemisphere, and also for many helpful discussions during the preparation of this paper. This work was supported by a C.A.S.E. award from the Science Research Council, in conjunction with the National Maritime Institute.

Appendix A

The wave source $\Psi_0^{(m)}$, defined in § 2, can be written as the sum of two terms,

$$\Psi_0^{(m)} = \Psi_{0,1}^{(m)} + \Psi_{0,2}^{(m)}, \tag{A 1}$$

where

$$\Psi_{0,1}^{(m)} = \cos m\psi \int_0^\infty \frac{k^{m+1} - K^{m+1}}{k-K} e^{-ky} J_m(kr)\, dk, \tag{A 2}$$

$$\Psi_{0,2}^{(m)} = \cos m\psi \int_0^\infty \frac{K^{m+1}}{k-K} e^{-ky} J_m(kr)\, dk. \tag{A 3}$$

By noting that

$$\frac{k^{m+1} - K^{m+1}}{k-K} = \sum_{n=0}^{m} k^n K^{m-n}, \tag{A 4}$$

and using the identity

$$\cos m\psi \int_0^\infty k^n e^{-ky} J_m(kr)\, dk = (-1)^m (n+m)! \frac{P_n^{-m}}{R^{n+1}}(\mu) \quad (\mu = \cos\theta), \tag{A 5}$$

it is easy to deduce that

$$\Psi_{0,1}^{(m)} = (-1)^m K^{m+1} \cos m\psi \sum_{n=1}^{m} \frac{(n+m)!}{(KR)^{n+1}} P_n^{-m}(\mu). \tag{A 6}$$

An expansion for the potential $\Psi_{0,2}^{(m)}$ can be found in a way similar to that used by Ursell (1963, § 3). We first use the identity

$$J_m(kr) = \frac{(i)^m}{2\pi} \int_0^{2\pi} e^{-ikr\cos\beta} \cos m\beta\, d\beta \tag{A 7}$$

to rewrite (A 3) as

$$\Psi_{0,2}^{(m)} = \frac{(i)^m}{2\pi} K^{m+1} \int_0^{2\pi} \cos m\beta \, d\beta \oint_0^{\infty} \frac{e^{-u\xi}}{u-1} du \cos m\psi, \tag{A 8}$$

where

$$\xi = K(y + ir\cos\beta), \quad y = R\cos\theta, \quad r = R\sin\theta. \tag{A 9}$$

Now consider the following Laplace transform:

$$\begin{aligned}\int_0^{\infty} e^{-\xi s} \oint_0^{\infty} \frac{e^{-u\xi}}{u-1} du \, d\xi &= \oint_0^{\infty} \frac{du}{(u-1)(u+s)} \\ &= [\pi i + \ln s](1+s)^{-1} \\ &= [\pi i + \ln s] \sum_{n=0}^{\infty} (-1)^n s^{-n-1} \quad (|s| > 1).\end{aligned}$$

But

$$\int_0^{\infty} e^{-s\xi} \left(\frac{\xi^{\nu}}{\nu!}\right) d\xi = s^{-\nu-1},$$

$$\int_0^{\infty} e^{-s\xi} \frac{\partial}{\partial \nu} \left(\frac{\xi^{\nu}}{\nu!}\right) d\xi = -(\ln s) s^{-\nu-1},$$

and so it follows that

$$\oint_0^{\infty} \frac{e^{-u\xi}}{u-1} du = \pi i \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial \nu} \left(\frac{\xi^{\nu}}{\nu!}\right)_{\nu=n}. \tag{A 10}$$

Returning to (A 8), we can use the result

$$\begin{aligned}\int_0^{2\pi} \cos m\beta \, \xi^{\nu} d\beta &= \int_0^{2\pi} \cos m\beta (\cos\theta + i\sin\theta\cos\beta)^{\nu} d\beta \, (KR)^{\nu} \\ &= (-i)^m 2\pi \frac{\nu!}{(\nu+m)!} (KR)^{\nu} P_{\nu}^m(\mu)\end{aligned} \tag{A 11}$$

(see Erdélyi *et al.* 1953, 3.7.25) to deduce that

$$\begin{aligned}\Psi_{0,2}^{(m)} = {} & \pi i K^{m+1} \sum_{n=m}^{\infty} \frac{(-1)^n}{(n+m)!} (KR)^n P_n^m(\mu) \cos m\psi \\ & - K^{m+1} \sum_{n=0}^{\infty} (-1)^n \frac{\partial}{\partial \nu} \left\{\frac{(KR)^{\nu}}{(\nu+m)!} P_{\nu}^m(\mu)\right\}_{\nu=n} \cos m\psi.\end{aligned} \tag{A 12}$$

This completes the expansion of $\Psi_0^{(m)}$ in terms of spherical harmonics.

Appendix B

We will now derive analytic expressions for the values of the integral

$$I\{\nu, \sigma; m\} = \int_0^1 P_{\nu}^m(\mu) P_{\sigma}^m(\mu) \, d\mu \quad (m = 0, 1, 2, \ldots). \tag{B 1}$$

The associated Legendre function $P_{\nu}^m(\mu)$ satisfies the equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_{\nu}^m}{d\mu}\right] + \left[\nu(\nu+1) - \frac{m^2}{1-\mu^2}\right] P_{\nu}^m = 0, \tag{B 2}$$

and using this it is a trivial matter to deduce that

$$(\nu-\sigma)(\nu+\sigma+1)I\{\nu,\sigma;m\} = \int_0^1 \left\{P_\nu^m(\mu)\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_\sigma^m}{d\mu}\right] - P_\sigma^m(\mu)\frac{d}{d\mu}\left[(1-\mu^2)\frac{dP_\nu^m}{d\mu}\right]\right\}d\mu \tag{B 3}$$

$$= \int_0^1 \frac{d}{d\mu}\left[(1-\mu^2)\left\{P_\nu^m(\mu)\frac{dP_\sigma^m}{d\mu} - P_\sigma^m(\mu)\frac{dP_\nu^m}{d\mu}\right\}\right]d\mu \tag{B 4}$$

$$= \left\{P_\sigma^m(\mu)\frac{dP_\nu^m}{d\mu} - P_\nu^m(\mu)\frac{dP_\sigma^m}{d\mu}\right\}_{\mu=0}. \tag{B 5}$$

If we now use the results

$$P_\nu^m(\mu)\big|_{\mu=0} = \frac{2^m}{\pi^{\frac{1}{2}}}\frac{(\frac{1}{2}\nu+\frac{1}{2}m-\frac{1}{2})!}{(\frac{1}{2}\nu-\frac{1}{2}m)!}\cos\left[\tfrac{1}{2}\pi(\nu+m)\right], \tag{B 6}$$

$$\left.\frac{dP_\nu^m}{d\mu}\right|_{\mu=0} = \frac{2^{m+1}}{\pi^{\frac{1}{2}}}\frac{(\frac{1}{2}\nu+\frac{1}{2}m)!}{(\frac{1}{2}\nu-\frac{1}{2}m-\frac{1}{2})!}\sin\left[\tfrac{1}{2}\pi(\nu+m)\right], \tag{B 7}$$

it follows that

$$I\{\nu,\sigma;m\} = \frac{2^{2m+1}}{\pi}\frac{1}{(\nu-\sigma)(\nu+\sigma+1)}\{A(\nu,\sigma;m)\sin\left[\tfrac{1}{2}\pi(\nu+m)\right]\cos\left[\tfrac{1}{2}\pi(\sigma+m)\right]$$

$$-A(\sigma,\nu;m)\cos\left[\tfrac{1}{2}\pi(\nu+m)\right]\sin\left[\tfrac{1}{2}\pi(\sigma+m)\right]\}, \tag{B 8}$$

where

$$A(\nu,\sigma;m) = \frac{(\frac{1}{2}\nu+\frac{1}{2}m)!\,(\frac{1}{2}\sigma+\frac{1}{2}m-\frac{1}{2})!}{(\frac{1}{2}\nu-\frac{1}{2}m-\frac{1}{2})!\,(\frac{1}{2}\sigma-\frac{1}{2}m)!}. \tag{B 9}$$

This is a generalization of the result stated by Erdélyi *et al.* (1953, 3.12.15) for the case $m = 0$.

As might be expected, the result (B 8) simplifies greatly for certain 'special' choices of the variables ν, σ, m. For the problem of the floating hemisphere we need only consider the cases $m = 0$ and $m = 1$, and we note the following useful results.

$m = 0$:

$$I\{\nu,2t;0\} = (-1)^t\frac{2}{\pi}\frac{(t-\frac{1}{2})!}{t!}\frac{1}{(\nu-2t)(\nu+2t+1)}\frac{(\frac{1}{2}\nu)!}{(\frac{1}{2}\nu-\frac{1}{2})!}\sin\tfrac{1}{2}\pi\nu, \tag{B 10}$$

$$I\{\nu,2t+1;0\} = -(-1)^t\frac{2}{\pi}\frac{(t+\frac{1}{2})!}{t!}\frac{1}{(\nu-2t-1)(\nu+2t+2)}\frac{(\frac{1}{2}\nu-\frac{1}{2})!}{(\frac{1}{2}\nu)!}\cos\tfrac{1}{2}\pi\nu, \tag{B 11}$$

$$I\{2s,2t+1;0\} = \frac{(-1)^{s+t+1}}{4^{s+t}}\frac{(2s)!\,(2t+1)!}{(2s-2t-1)(2s+2t+2)}\frac{1}{\{s!\,t!\}^2}, \tag{B 12}$$

and the orthogonality properties

$$I\{2s,2t;0\} = \delta_{st}\frac{1}{4s+1}, \tag{B 13}$$

$$I\{2s+1,2t+1;0\} = \delta_{st}\frac{1}{4s+3}. \tag{B 14}$$

$m = 1$:

$$I\{\nu,2t;1\} = (-1)^t\frac{8}{\pi}\frac{(t+\frac{1}{2})!}{t!}\frac{t}{(\nu-2t)(\nu+2t+1)}\frac{(\frac{1}{2}\nu)!}{(\frac{1}{2}\nu-\frac{1}{2})!}\sin\tfrac{1}{2}\pi\nu, \tag{B 15}$$

$$I\{\nu, 2t+1; 1\} = -(-1)^t \frac{4(t+\frac{1}{2})!}{\pi t!} \frac{\nu}{(\nu-2t-1)(\nu+2t+2)} \frac{(\frac{1}{2}\nu+\frac{1}{2})!}{(\frac{1}{2}\nu)!} \cos\tfrac{1}{2}\pi\nu, \quad \text{(B 16)}$$

$$I\{2s+1, 2t; 1\} = \frac{(-1)^{s+t}}{4^{s+t}} \frac{t(2s+1)!\,(2t+1)!}{(2s-2t+1)(s+t+1)} \frac{1}{\{s!\,t!\}^2}, \quad \text{(B 17)}$$

and the orthogonality properties

$$I\{2s, 2t; 1\} = \delta_{st} \frac{2s(2s+1)}{4s+1}, \quad \text{(B 18)}$$

$$I\{2s+1, 2t+1; 1\} = \delta_{st} \frac{2(s+1)(2s+1)}{4s+3}. \quad \text{(B 19)}$$

REFERENCES

Barakat, R. 1962 Vertical motion of a floating sphere in a sine-wave sea. *J. Fluid Mech.* **13**, 540–556.

Davis, A. M. J. 1971 Short surface waves due to an oscillating half-immersed sphere. *Mathematika* **18**, 20–39.

Erdélyi, A., Magnus, W., Oberhettinger, F. & Tricomi, F. G. 1953 *Higher Transcendental Functions*, vol. 1. McGraw-Hill.

Greenhow, M. J. L. 1980 The hydrodynamic interactions of spherical wave power devices in surface waves. In *Power from Sea Waves* (ed. B. Count), pp. Academic.

Havelock, T. H. 1929 Forced surface waves on water. *Phil. Mag.* **8** (57), 569–576.

Havelock, T. 1955 Waves due to a floating hemi-sphere making periodic heaving oscillations. *Proc. R. Soc. Lond.* A **231**, 1–7.

Kim, W. D. 1965 On the harmonic oscillations of a rigid body on a free surface. *J. Fluid Mech.* **21**, 427–451.

Kotik, J. & Mangulis, V. 1962 On the Kramers Kronig relations for ship motions. *Int. Shipbuilding Prog.* **9**, 3–10.

Leppington, F. G. 1973 On the radiation and scattering of short surface waves. Part 3. *J. Fluid Mech.* **59**, 147–157.

Rhodes-Robinson, P. F. 1971 On the short cylindrical waves due to a body heaving on water. *Proc. Camb. Phil. Soc.* **70**, 311–321.

Riesz, F. 1913 *Systèmes d'Equations Linéaires à une Infinité d'Inconnues*. Gauthier-Villars.

Thorne, R. C. 1953 Multipole expansions in the theory of surface waves. *Proc. Camb. Phil. Soc.* **49**, 707–716.

Ursell, F. 1949 On the heaving motion of a circular cylinder on the surface of a fluid. *Quart. Jl Mech. Appl. Math.* **2**, 218–231.

Ursell, F. 1953 Short surface waves due to an oscillating immersed body. *Proc. R. Soc. Lond.* A **220**, 90–103.

Ursell, F. 1954 Water waves generated by oscillating bodies. *Quart. Jl Mech. Appl. Math.* **7**, 427–437.

Ursell, F. 1957 On the virtual mass and damping of floating bodies at zero speed ahead. In *Proc. Symp. Behaviour of Ships in a Survey, Wageningen.*

Ursell, F. 1963 The periodic heaving motion of a half-immersed sphere: The analytic form of the velocity potential and long-wave asymptotics of the virtual mass coefficient. *Dept of Mathematics, Univ. Manchester, Internal Rep.*